Existence of a spanning tree having small diameter

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Abstract

In this paper, we prove that for a sufficiently large integer \(d\) and a connected graph \(G\), if \(|V(G)| < \frac{(d+2)(d(G)+1)}{3}\), then there exists a spanning tree \(T\) of \(G\) such that \(\text{diam}(T) \leq d\).

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1 Introduction

In this paper, we consider only finite undirected simple graphs. Let \(G\) be a graph. We let \(V(G)\) and \(E(G)\) denote the vertex set and the edge set of \(G\), respectively. For \(x \in V(G)\), we let \(N_G(x)\) and \(N_G[x]\) denote the (open) neighborhood and the closed neighborhood of \(x\), respectively; thus \(N_G(x) = \{y \in V(G) : xy \in E(G)\}\) and \(N_G[x] = N_G(x) \cup \{x\}\). For \(X \subseteq V(G)\), we let \(G[X]\) denote the subgraph of \(G\) induced by \(X\). Let \(G\) be a connected graph. For \(x, y \in V(G)\), the distance between \(x\) and \(y\), denoted by \(d_G(x,y)\), is the minimum length of a path connecting \(x\) and \(y\).

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For $X \subseteq V(G)$ and a vertex $y \in V(G)$, we let $d_G(X, y) = \min \{d_G(x, y) : x \in X\}$. For $X \subseteq V(G)$ and $i \in \mathbb{N} \cup \{0\}$, let $N_G^i(X) = \{y \in V(G) : d_G(X, y) = i\}$. For $x \in V(G)$, we let $N_G^0(x) = N_G^1(\{x\})$; thus $N_G^0(x) = \{x\}$ and $N_G^1(x) = N_G^2(x)$. For $x \in V(G)$, the eccentricity of $x$, denoted by $\text{ecc}_G(x)$, is defined as the maximum of $d_G(x, y)$ as $y$ ranges over $V(G)$. The diameter of $G$, denoted by $\text{diam}(G)$, is defined by $\text{diam}(G) = \max \{\text{ecc}_G(x) : x \in V(G)\}$, and the radius of $G$, denoted by $\text{rad}(G)$, is defined by $\text{rad}(G) = \min \{\text{ecc}_G(x) : x \in V(G)\}$. A vertex $x$ of $G$ is central if $\text{ecc}_G(x) = \text{rad}(G)$. For terms and symbols not defined here, we refer the reader to [1].

The purpose of this paper is to give a sufficient condition for the existence of a spanning tree having small diameter. Our primary motivation derives from a minimum diameter spanning tree problem. For a connected graph $G$, a minimum diameter spanning tree (or MDST) of $G$ is a spanning tree having minimum diameter among all possible spanning trees of $G$. From the standpoint of applications to a vehicle routing problem, MDST is widely studied in combinatorial optimization theory. For example, for a given connected graph $G = (V, E)$, an MDST of $G$ can be computed in $O(|E||V| \log |V|)$ time (see Theorem 7.4 in [5]). On the other hand, there are few results which give an explicit upper bound of the diameter of an MDST as our main theorems do.

Our second motivation is to give a refinement of a known result concerning the relationship between the diameter and the minimum degree of a graph. In [2], Erdős, Pach, Pollack and Tuza implicitly proved the following theorem.

**Theorem A (Erdős, Pach, Pollack and Tuza [2])** Let $G$ be a connected graph with $\delta(G) \geq 2$. If $|V(G)| \leq \frac{(d+1)(\delta(G)+1)}{3}$, then $\text{diam}(G) \leq d$.

Note that even if $G$ is a connected graph with $\text{diam}(G) \leq d$, it is not always true that $G$ has a spanning tree $T$ with $\text{diam}(T) \leq d$. Thus Theorem A gives us no useful information about spanning trees having small diameter.

Recently, Kano and Matsumura [3] posed the following conjecture and proved that the conjecture is true if $d$ is even or $d \in \{5, 7, 9\}$.

**Conjecture 1** Let $d \geq 4$ be an integer, and let $G$ be a connected graph. If $|V(G)| \leq \frac{(d+1)(\delta(G)+1)}{3}$, then there exists a spanning tree $T$ of $G$ such that $\text{diam}(T) \leq d$.

Now we give a conjecture which is slightly stronger than Conjecture 1.

**Conjecture 2** Let $d \geq 4$ be an integer, and let $G$ be a connected graph. If $|V(G)| < \frac{(d+2)(\delta(G)+1)}{3}$, then there exists a spanning tree $T$ of $G$ such that $\text{diam}(T) \leq d$.  

2
Note that if a connected graph $G$ has a spanning tree $T$ with $\text{diam}(T) \leq d$, then $\text{diam}(G) \leq d$. Thus, if Conjecture 2 is true, then it is a refinement of Theorem A.

Now we discuss the sharpness of Conjecture 2. Let $d \geq 4$ and $m \geq 1$ be integers. Let $H_1, H_2, \ldots, H_{d+2}$ be vertex-disjoint complete graphs of order $m$, and let $G_{d,m}$ be the graph obtained from the union of $H_1, H_2, \ldots, H_{d+2}$ by joining all vertices of $H_i$ to all vertices of $H_{i+1}$ for every $i$ $(1 \leq i \leq d+2)$, where $H_{d+3} = H_1$. Then $|V(G_{d,m})| = (d+2)m = \frac{(d+2)(\delta(G)+1)}{4}$. Furthermore, in [3], Kano and Matsumura showed that $G_{d,m}$ has no spanning tree $T$ such that $\text{diam}(T) \leq d$. Thus, if Conjecture 2 is true, then the condition concerning the order of the graph is sharp.

In this paper, we prove that Conjecture 2 is true for almost all $d$. Our main theorem is the following.

**Theorem 1.1** Let $d$ be an integer with

$$d \geq \begin{cases} 4 & \text{(if } d \equiv 0 \pmod{2}) \\ 7 & \text{(if } d \equiv 1 \pmod{6}) \\ 39 & \text{(if } d \equiv 3 \pmod{6}) \\ 41 & \text{(if } d \equiv 5 \pmod{6}), \end{cases}$$

and let $G$ be a connected graph. If $|V(G)| < \frac{(d+2)(\delta(G)+1)}{3}$, then there exists a spanning tree $T$ of $G$ such that $\text{diam}(T) \leq d$.

On the other hand, we prove that Conjecture 2 is also true for small values of $d$ as follows.

**Theorem 1.2** Let $d \in \{5, 9\}$, and let $G$ be a connected graph. If $|V(G)| < \frac{(d+2)(\delta(G)+1)}{4}$, then there exists a spanning tree $T$ of $G$ such that $\text{diam}(T) \leq d$.

We prove Theorem 1.1 for the case where $d$ is even in Section 2. In Section 3, we prepare some lemmas used in the proof of Theorem 1.1 for the case where $d$ is odd, and prove Theorem 1.1 for the case where $d \equiv 1 \pmod{6}$. In Section 4, we prove the remaining cases of Theorem 1.1. In Section 5, we focus on small $d$, and prove Theorem 1.2.

In the proof of Theorem 1.1, we make use of the following well-known lemmas.

**Lemma 1.3** Let $T$ be a tree. Then $2\text{rad}(T) - 1 \leq \text{diam}(T) \leq 2\text{rad}(T)$.

**Lemma 1.4** Let $r \geq 1$ be an integer, and let $G$ be a connected graph.

(i) If there exists a vertex $a_1 \in V(G)$ such that $d_G(a_1, x) \leq r$ for all $x \in V(G)$, then $G$ has a spanning tree $T$ such that $\text{diam}(T) \leq 2r$. 

3
(ii) If there exists an edge $a_1a_2 \in E(G)$ such that $d_G(\{a_1, a_2\}, x) \leq r$ for all $x \in V(G)$, then $G$ has a spanning tree $T$ such that $\text{diam}(T) \leq 2r + 1$.

2 The case $d \equiv 0 \pmod{2}$

In this section, we show that Theorem 1.1 holds for the case where $d \equiv 0 \pmod{2}$. The following theorem was proved by Kim, Rho, Song and Hwang [4].

**Theorem B (Kim, Rho, Song and Hwang [4])** Let $G$ be a connected graph. If $\delta(G) \geq 2$ and $\text{rad}(G) \geq 3$, then $\text{rad}(G) \leq \frac{3|V(G)|}{2(\delta(G)+1)}$.

**Lemma 2.1** Let $d \geq 4$ be an integer, and let $G$ be a connected graph. If $|V(G)| < \frac{(d+2)(\delta(G)+1)}{3}$, then $\text{rad}(G) \leq \frac{d+1}{2}$.

**Proof.** For the moment, assume that $\delta(G) = 1$. Then $|V(G)| < \frac{(d+2)(\delta(G)+1)}{3} = \frac{2(d+2)}{3} \leq d+1$. This implies that the length of any path of $G$ is at most $d-1$. Hence for a spanning tree $T$ of $G$, it follows from Lemma 1.3 that $2\text{rad}(T)-1 \leq \text{diam}(T) \leq d-1$ (i.e., $\text{rad}(T) \leq \frac{d}{2}$). Since the deletion of edges cannot decrease the radius, it follows that $\text{rad}(G) \leq \frac{d}{2}$, as desired. Thus we may assume that $\delta(G) \geq 2$.

Since $d \geq 4$, if $\text{rad}(G) \leq 2$, then the desired conclusion clearly holds. Thus we may assume that $\text{rad}(G) \geq 3$. Then it follows from Theorem B that

$$\text{rad}(G) \leq \frac{3|V(G)|}{2(\delta(G)+1)} < \frac{3 \cdot (d+2)(\delta(G)+1)}{2(\delta(G)+1)} = \frac{d+2}{2},$$

as desired. ■

Theorem 1.1 for the case where $d \equiv 0 \pmod{2}$ immediately follows from Lemma 2.1.

**Lemma 2.2** If $d \geq 4$ is an even integer, then Theorem 1.1 holds.

**Proof.** Let $G$ be a connected graph such that $|V(G)| < \frac{(d+2)(\delta(G)+1)}{3}$. Since $d$ is even, it follows from Lemma 2.1 that $\text{rad}(G) \leq \frac{d}{2}$. Thus a central vertex $a$ of $G$ satisfies $d_G(a,x) \leq \frac{d}{2}$ for all $x \in V(G)$. Hence by Lemma 1.4(i), $G$ has a spanning tree $T$ such that $\text{diam}(T) \leq 2 \cdot \frac{d}{2} = d$. ■

3 Lemmas

Throughout this section, fix an odd integer $d \geq 5$, and write $d = 6k + \alpha$ ($k \in \mathbb{N} \cup \{0\}, \alpha \in \{1,3,5\}$). A set $X$ of vertices of a graph $G$ is $d$-good if

- **(G1)** either $|X| = 1$, or $|X| = 2$ and the vertices in $X$ are adjacent, and
\((G2)\) \[ d_G(X, x) \leq 3k + \frac{\alpha - 1}{2} \] for all \(x \in V(G)\).

**Lemma 3.1** If a graph \(G\) has a \(d\)-good set, then \(G\) has a spanning tree \(T\) such that \(\text{diam}(T) \leq d\).

**Proof.** If \(\{a\}\) is \(d\)-good for a vertex \(a \in V(G)\), then by Lemma 1.4(i), \(G\) has a spanning tree \(T\) with \(\text{diam}(T) \leq 2(3k + \frac{\alpha - 1}{2}) < d\); if \(\{a, b\}\) is \(d\)-good for two vertices \(a, b \in V(G)\) with \(ab \in E(G)\), then by Lemma 1.4(ii), \(G\) has a spanning tree \(T\) with \(\text{diam}(T) \leq 2(3k + \frac{\alpha - 1}{2}) + 1 = d\). In either case, we obtain the desired conclusion.

In the rest of this section, fix a connected graph \(G\) with \(|V(G)| < \frac{(d+2)(\delta(G)+1)}{3} = (2k + \frac{\alpha+2}{2})(\delta(G) + 1)\) and a central vertex \(a_0\) of \(G\). Set \(\delta := \delta(G)\), and for \(i \geq 0\), let \(X_i := N_G^i(a_0)\). By Lemma 2.1, \(\text{rad}(G) \leq \frac{d+1}{2} = 3k + \frac{\alpha+1}{2}\). Thus the following lemma holds.

**Lemma 3.2** For \(i \geq 3k + \frac{\alpha+3}{2}\), \(X_i = \emptyset\).

**Lemma 3.3** Let \(h\) and \(i_0\) be integers with \(h \geq 2\) and \(h + 1 \leq i_0 \leq 3k + \frac{\alpha+1}{2} - h\). If there exists a vertex \(a_{i_0} \in X_{i_0}\) such that \(d_G(a_{i_0}, x) \leq h\) for all \(x \in X_{i_0}\), then \(G\) has a \(d\)-good set.

**Proof.** Let \(a_0a_1 \cdots a_{i_0}\) be a shortest path joining \(a_0\) and \(a_{i_0}\). Note that \(a_i \in X_i\) for each \(i\) (\(1 \leq i \leq i_0 - 1\)). We prove that \(\{a_h, a_{h+1}\}\) is \(d\)-good. Let \(x \in V(G)\), and let \(i_1\) be the index such that \(x \in X_{i_1}\). It suffices to show that \(d_G(\{a_h, a_{h+1}\}, x) \leq 3k + \frac{\alpha - 1}{2}\). If \(i_1 \leq i_0 - 1\), then

\[
d_G(a_h, x) \leq d_G(a_h, a_0) + d_G(a_0, x) = h + i_1 \leq h + (i_0 - 1) \leq 3k + \frac{\alpha - 1}{2},
\]

as desired. Thus we may assume that \(i_1 \geq i_0\). Then there exists a vertex \(x' \in X_{i_0}\) such that \(d_G(x', x) = i_1 - i_0\). Since \(i_1 \leq 3k + \frac{\alpha+1}{2}\) by Lemma 3.2, it follows that

\[
d_G(a_{h+1}, x) \leq d_G(a_{h+1}, a_{i_0}) + d_G(a_{i_0}, x') + d_G(x', x) \\
\leq (i_0 - (h + 1)) + h + (i_1 - i_0) \\
\leq i_1 - 1 \\
\leq 3k + \frac{\alpha - 1}{2},
\]

as desired.

**Lemma 3.4** For \(i\) \((3 \leq i \leq 3k + \frac{\alpha-3}{2})\), if \(|X_{i-1} \cup X_i \cup X_{i+1}| \leq 2\delta + 1\), then \(G\) has a \(d\)-good set.
Proof. If \( X_i = \emptyset \), then \( \{a_0\} \) is a \( d \)-good set. Thus we may assume that \( X_i \neq \emptyset \). Let \( a_i \in X_i \). For each \( x \in X_i \), since \( N_G[a_i] \cup N_G[x] \subseteq X_{i-1} \cup X_i \cup X_{i+1} \),

\[
2 \delta + 1 \geq |X_{i-1} \cup X_i \cup X_{i+1}|
\]

\[
\geq |N_G[a_i]| + |N_G[x]| - |N_G[a_i] \cap N_G[x]|
\]

\[
\geq 2(\delta + 1) - |N_G[a_i] \cap N_G[x]|,
\]

and hence \( N_G[a_i] \cap N_G[x] \neq \emptyset \). In particular, \( d_G(a_i, x) \leq 2 \) for each \( x \in X_i \). Thus, applying Lemma 3.3 with \( h = 2 \), we see that \( G \) has a \( d \)-good set. \( \Box \)

Lemma 3.5 If \( \alpha = 1 \), then Theorem 1.1 holds.

Proof. Note that \( |V(G)| < (2k + 1)(\delta + 1) \). By Lemma 3.1, it suffices to show that \( G \) has a \( d \)-good set. By way of contradiction, suppose that \( G \) has no \( d \)-good set (i.e., for any vertices \( a, b \in V(G) \) with \( a = b \) or \( ab \in E(G) \), there exists a vertex \( x \in V(G) \) such that \( d_G(\{a, b\}, x) \geq 3k + 1 \)). Since \( \{a_0\} \) is not \( d \)-good, we have \( X_{3k+1} \neq \emptyset \). Let \( a_{3k} \in X_{3k} \), and let \( a_0a_1 \cdots a_{3k} \) be a shortest path joining \( a_0 \) and \( a_{3k} \). Note that \( a_i \in X_i \) for each \( i \) \((1 \leq i \leq 3k - 1) \). By Lemmas 3.2 and 3.4,

\[
|V(G)| = |N_G[a_0]| + \sum_{1 \leq j \leq k} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}|
\]

\[
\geq (\delta + 1) + (k - 1)(2\delta + 2) + |X_{3k-1} \cup X_{3k} \cup X_{3k+1}|.
\]

Since \( |V(G)| < (2k + 1)(\delta + 1) \), this implies that \( |X_{3k-1} \cup X_{3k} \cup X_{3k+1}| \leq 2\delta + 1 \). Hence

\[
d_G(a_{3k}, x) \leq 2 \text{ for all } x \in X_{3k}.
\] (3.1)

Since \( \{a_2, a_3\} \) is not \( d \)-good, there exists a vertex \( x^* \in V(G) \) such that \( d_G(\{a_2, a_3\}, x^*) \geq 3k + 1 \). Let \( i \) be the index such that \( x^* \in X_i \). Since

\[
3k + 1 \leq d_G(a_2, x^*) \leq d_G(a_2, a_0) + d_G(a_0, x^*) = 2 + i,
\]

we have \( 3k - 1 \leq i \leq 3k + 1 \). If \( N_G[x^*] \cap X_{3k} \neq \emptyset \), say \( y \in N_G[x^*] \cap X_{3k} \), then by (3.1),

\[
d_G(a_3, x^*) \leq d_G(a_3, a_{3k}) + d_G(a_{3k}, y) + d_G(y, x^*) \leq (3k - 3) + 2 + 1 = 3k,
\]

which is a contradiction. Thus

\[
N_G[x^*] \cap X_{3k} = \emptyset.
\] (3.2)

In particular, \( i = 3k - 1 \). Let \( a_0b_1 \cdots b_{3k-1} \) be a shortest path of \( G \) joining \( a_0 \) and \( x^* \) where \( b_{3k-1} = x^* \). Note that \( b_i \in X_i \) for each \( i \) \((1 \leq i \leq 3k - 2) \).
Let \( c \in X_{3k+1} \). We now focus on the set \( A = \{c\} \cup \{a_{3j-2} : 1 \leq j \leq k\} \cup \{b_{3j-1} : 1 \leq j \leq k\} \) (see Figure 1). Note that \(|A| = 2k+1\). Since \(|V(G)| < (2k+1)(\delta+1)\) and \(\sum_{u \in A} |N_G[u]| \geq |A| (\delta + 1) = (2k+1)(\delta + 1)\), there exist two vertices \( u, v \in A \) such that \( N_G[u] \cap N_G[v] \neq \emptyset \). This together with (3.2) implies that one of the following holds:

(A1) \( N_G[a_{3j+1}] \cap N_G[b_{3j-1}] \neq \emptyset \) for some \( j \) (1 \( \leq j \leq k \)),

(A2) \( N_G[a_{3j-2}] \cap N_G[b_{3j-1}] \neq \emptyset \) for some \( j \) (2 \( \leq j \leq k \)), or

(A3) \( N_G[a_1] \cap N_G[b_2] \neq \emptyset \).

If (A1) holds, then
\[
d_G(a_3, x^*) \leq d_G(a_3, a_{3j+1}) + d_G(a_{3j+1}, b_{3j-1}) + d_G(b_{3j-1}, b_{3k-1})
\leq ((3j + 1) - 3) + 2 + ((3k - 1) - (3j - 1))
= 3k;
\]

if (A2) holds, then
\[
d_G(a_3, x^*) \leq d_G(a_3, a_{3j-2}) + d_G(a_{3j-2}, b_{3j-1}) + d_G(b_{3j-1}, b_{3k-1})
\leq ((3j - 2) - 3) + 2 + ((3k - 1) - (3j - 1))
= 3k - 3;
\]

if (A3) holds, then
\[
d_G(a_2, x^*) \leq d_G(a_2, a_1) + d_G(a_1, b_2) + d_G(b_2, b_{3k-1})
\leq 1 + 2 + ((3k - 1) - 2)
= 3k.
\]

In any case, we obtain a contradiction. \( \blacksquare \)
Lemma 3.6 Suppose that $\alpha \in \{3, 5\}$, $k \geq 1$, and there exist vertices $a_2, b_2 \in X_2$ such that for each $i$ ($3k + \frac{\alpha - 5}{2} \leq i \leq 3k + \frac{\alpha + 1}{2}$), we have $d_G(\{a_2, b_2\}, x) = i - 2$ for all $x \in X_i$ (here $a_2$ may be equal to $b_2$). Then $G$ has a $d$-good set.

Proof. By way of contradiction, suppose that $G$ has no $d$-good set (i.e., for any vertices $a, b \in V(G)$, if $a = b$ or $ab \in E(G)$, then there exists a vertex $x \in V(G)$ such that $d_G(\{a, b\}, x) \geq 3k + \frac{\alpha + 1}{2}$). Since $\{a_0\}$ is not $d$-good, we have $X_{3k + \frac{\alpha + 1}{2}} \neq \emptyset$. By Lemma 3.2, $X_{3k + \frac{\alpha + 1}{2}} = \emptyset$.

For the moment, we suppose that $d_G(\{a_2, b_2\}) \leq 3$. Then there exist vertices $a'_2 \in N_G[a_2]$ and $b'_2 \in N_G[b_2]$ such that $a'_2 = b'_2$ or $a'_2 b'_2 \in E(G)$. For $x \in V(G)$, if $x \in \bigcup_{3k + \frac{\alpha - 5}{2} \leq i \leq 3k + \frac{\alpha + 1}{2}} X_i$, then by the assumption of the lemma,

$$d_G(\{a'_2, b'_2\}, x) \leq 1 + d_G(\{a_2, b_2\}, x) \leq 3k + \frac{\alpha - 1}{2};$$

if $x \in \bigcup_{0 \leq i \leq 3k + \frac{\alpha - 5}{2}} X_i$, then

$$d_G(\{a'_2, b'_2\}, x) \leq 1 + d_G(\{a_2, b_2\}, a_0) + d_G(a_0, x) \leq 1 + 2 + \left(3k + \frac{\alpha - 7}{2}\right).$$

Since $x \in V(G)$ is arbitrary, this implies that $\{a'_2, b'_2\}$ is $d$-good, which is a contradiction. Thus

$$d_G(\{a_2, b_2\}) \geq 4. \quad (3.3)$$

Let $a_1 \in N_G(a_2) \cap X_1$ and $b_1 \in N_G(b_2) \cap X_1$.

Claim 3.1 If $|N_G(a_1) \cap N_G(b_1)| < \frac{\delta + 1}{3}$, then $|X_0 \cup X_1 \cup X_2| > \frac{5(\delta + 1)}{3}$.

Proof. Since $a_1 \neq b_1$ and $a_1 b_1 \notin E(G)$ by (3.3),

$$|X_0 \cup X_1 \cup X_2| \geq |N_G[a_1] \cup N_G[b_1]|$$

$$= |N_G[a_1]| + |N_G[b_1]| - |N_G[a_1] \cap N_G(b_1)|$$

$$> 2(\delta + 1) - \frac{\delta + 1}{3},$$

as desired. ■

Claim 3.2 If $|N_G(a_1) \cap N_G(b_1)| \geq \frac{\delta + 1}{3}$, then $|\bigcup_{0 \leq i \leq 3} X_i| \geq \frac{7(\delta + 1)}{3}$.

Proof. Since $N_G[a_2], N_G[b_2]$ and $N_G(a_1) \cap N_G(b_1)$ are pairwise disjoint by (3.3),

$$\left| \bigcup_{0 \leq i \leq 3} X_i \right| \geq |N_G[a_2]| + |N_G[b_2]| + |N_G(a_1) \cap N_G(b_1)| \geq 2(\delta + 1) + \frac{\delta + 1}{3},$$

as desired. ■

Recall that $|V(G)| < (2k + \frac{\alpha + 2}{3})(\delta + 1)$ and $X_{3k + \frac{\alpha + 3}{2}} = \emptyset$. 8
Claim 3.3 Either $\alpha = 3$ and $|N_G(a_1) \cap N_G(b_1)| \geq \frac{5+1}{3}$, or $\alpha = 5$ and $|N_G(a_1) \cap N_G(b_1)| < \frac{5+1}{3}$.

Proof. By way of contradiction, suppose that either $\alpha = 3$ and $|N_G(a_1) \cap N_G(b_1)| < \frac{5+1}{3}$, or $\alpha = 5$ and $|N_G(a_1) \cap N_G(b_1)| \geq \frac{5+1}{3}$. By Lemma 3.4, $|X_{i-1} \cup X_i \cup X_{i+1}| \geq 2(\delta + 1)$ for all $i$ ($3 \leq i \leq 3k + \frac{a-3}{2}$). Hence, if $\alpha = 3$, then by Claim 3.1,

$$\left(2k + \frac{5}{3}\right)(\delta + 1) > |V(G)|$$

$$= \left|X_0 \cup X_1 \cup X_2\right| + \sum_{1 \leq j \leq k} |X_{3j} \cup X_{3j+1} \cup X_{3j+2}|$$

$$\geq \frac{5(\delta + 1)}{3} + 2(k-1)(\delta + 1) + |X_{3k} \cup X_{3k+1} \cup X_{3k+2}|;$$

if $\alpha = 5$, then by Claim 3.2,

$$\left(2k + \frac{7}{3}\right)(\delta + 1) > |V(G)|$$

$$= \left|\bigcup_{0 \leq i \leq 3} X_i\right| + \sum_{1 \leq j \leq k} |X_{3j+1} \cup X_{3j+2} \cup X_{3j+3}|$$

$$\geq \frac{7(\delta + 1)}{3} + 2(k-1)(\delta + 1) + |X_{3k+1} \cup X_{3k+2} \cup X_{3k+3}|.$$

In either case, we obtain

$$\left|X_{3k+\frac{a-3}{2}} \cup X_{3k+\frac{a-1}{2}} \cup X_{3k+\frac{a+1}{2}}\right| < 2(\delta + 1).$$

This implies that $N_G[x] \cap N_G[x'] \neq \emptyset$ for any $x, x' \in X_{3k+\frac{a-3}{2}} \cup X_{3k+\frac{a+1}{2}}$.

Fix a vertex $x^* \in X_{3k+\frac{a-1}{2}}$. Since $d_G(\{a_2, b_2\}, x^*) = 3k + \frac{a-5}{2}$, we may assume that $d_G(a_2, x^*) = 3k + \frac{a-5}{2}$. For $x \in V(G)$, if $x \in X_{3k+\frac{a-1}{2}} \cup X_{3k+\frac{a+1}{2}}$, then $d_G(a_2, x) = d_G(a_2, x^*) + d_G(x^*, x) \leq (3k + \frac{a-5}{2}) + 2$; if $x \in V(G) - (X_{3k+\frac{a-1}{2}} \cup X_{3k+\frac{a+1}{2}})$, then $d_G(a_1, x) \leq d_G(a_1, a_0) + d_G(a_0, x) \leq 1 + (3k + \frac{a-3}{2})$. Thus $d_G(\{a_1, a_2\}, x) \leq 3k + \frac{a-1}{2}$ for all $x \in V(G)$. Hence $\{a_1, a_2\}$ is $d$-good, which is a contradiction. ■

By Lemma 3.4, $|X_{i-1} \cup X_i \cup X_{i+1}| \geq 2(\delta + 1)$ for all $i$ ($3 \leq i \leq 3k + \frac{a-3}{2}$). Hence, if $\alpha = 3$, then

$$\left(2k + \frac{5}{3}\right)(\delta + 1) > |V(G)|$$

$$= |N_G[a_0]| + \sum_{1 \leq j \leq k} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}| + |X_{3k+2}|$$

$$\geq (\delta + 1) + 2k(\delta + 1) + |X_{3k+2}|;$$
if $\alpha = 5$, then by Claims 3.1 and 3.3,
\[
\left(2k + \frac{7}{3}\right) (\delta + 1) > |V(G)|
\]
\[
= |X_0 \cup X_1 \cup X_2| + \sum_{1 \leq j \leq k} |X_{3j} \cup X_{3j+1} \cup X_{3j+2}| + |X_{3k+3}|
\geq \frac{5(\delta + 1)}{3} + 2k(\delta + 1) + |X_{3k+3}|.
\]
In either case, we obtain
\[
|X_{3k+\frac{\alpha-1}{2}}| < \frac{2(\delta + 1)}{3}.
\] (3.4)

Furthermore, if $\alpha = 3$, then by Claims 3.2 and 3.3,
\[
\left(2k + \frac{5}{3}\right) (\delta + 1) > |V(G)|
\]
\[
= |N_G[a_0]| + \sum_{1 \leq j \leq k} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}| + |X_{3k+2} \cup X_{3k+3}|
\geq (\delta + 1) + 2k(\delta + 1) + |X_{3k+2} \cup X_{3k+3}|.
\]

if $\alpha = 5$, then
\[
\left(2k + \frac{7}{3}\right) (\delta + 1) > |V(G)|
\]
\[
= |N_G[a_0]| + \sum_{1 \leq j \leq k} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}| + |X_{3k+2} \cup X_{3k+3}|
\geq (\delta + 1) + 2k(\delta + 1) + |X_{3k+2} \cup X_{3k+3}|.
\]

In either case, we obtain
\[
|X_{3k+\frac{\alpha-1}{2}} \cup X_{3k+\frac{\alpha+1}{2}}| < \frac{4(\delta + 1)}{3}.
\] (3.5)

It follows from (3.4) and (3.5) that
\[
\delta + 1 = \frac{2(\delta + 1)}{3} + \frac{\delta + 1}{3} > \frac{|X_{3k+\frac{\alpha-1}{2}} \cup X_{3k+\frac{\alpha+1}{2}}|}{2} + \frac{|X_{3k+\frac{\alpha+1}{2}}}{2}.
\]

This implies that
\[
|N_G[x] \cap X_{3k+\frac{\alpha-1}{2}}| \geq (\delta + 1) - \frac{|X_{3k+\frac{\alpha+1}{2}}|}{2} > \frac{|X_{3k+\frac{\alpha-1}{2}}}{2} \text{ for all } x \in X_{3k+\frac{\alpha+1}{2}}.
\] (3.6)

Recall that $d_G(a_2, b_2, x) = 3k + \frac{\alpha-5}{2}$ for all $x \in X_{3k+\frac{\alpha-1}{2}}$. By the symmetry of $a_2$ and $b_2$, we may assume that
\[
\left| \left\{ x \in X_{3k+\frac{\alpha-1}{2}} : d_G(a_2, x) = 3k + \frac{\alpha-5}{2} \right\} \right| \geq \frac{|X_{3k+\frac{\alpha-1}{2}}|}{2}.
\] (3.7)
By (3.6) and (3.7), each vertex \( x \in X_{3k+\frac{\alpha+1}{2}} \) is adjacent to a vertex \( y_x \in X_{3k+\frac{\alpha+1}{2}} \) such that \( d_G(a_2, y_x) = 3k + \frac{\alpha-5}{2} \). Thus for \( x \in V(G) \), if \( x \in X_{3k+\frac{\alpha+1}{2}} \), then
\[
d_G(a_1, x) \leq d_G(a_1, a_2) + d_G(a_2, y_x) + d_G(y_x, x) = 1 + \left( 3k + \frac{\alpha-5}{2} \right) + 1;
\]
if \( x \in V(G) - X_{3k+\frac{\alpha+1}{2}} \), then \( d_G(a_0, x) \leq 3k + \frac{\alpha-1}{2} \). Consequently \( d_G({a_0, a_1}, x) \leq 3k + \frac{\alpha-1}{2} \) for all \( x \in V(G) \). Therefore \( \{a_0, a_1\} \) is \( d \)-good, which is a contradiction.

This completes the proof of Lemma 3.6. ■

4 The case \( d \equiv 1 \pmod{2} \)

In this section, we complete the proof of Theorem 1.1. Let \( d \) and \( G \) be as in Theorem 1.1, and write \( d = 6k + \alpha \) (\( k \in \mathbb{N} \cup \{0\} \), \( 0 \leq \alpha \leq 5 \)). By Lemmas 2.2 and 3.5, we may assume that \( \alpha \in \{3, 5\} \). Note that
\[
|V(G)| < \begin{cases} (2k + \frac{5}{3})(\delta + 1) & (\alpha = 3) \\ (2k + \frac{7}{3})(\delta + 1) & (\alpha = 5) \end{cases}
\]
and \( k \geq 6 \). Let \( a_0 \) be a central vertex of \( G \). Set \( \delta := \delta(G) \) and for \( i \geq 0 \), let \( X_i := N_{k^i}(a_0) \).

By way of contradiction, suppose that \( G \) has no spanning tree \( T \) such that \( \text{diam}(T) \leq d \). Then by Lemma 3.1, \( G \) has no \( d \)-good set. Since \( \{a_0\} \) is not \( d \)-good, we have \( X_{3k+\frac{\alpha+1}{2}} \neq \emptyset \). By Lemma 3.2, \( X_{3k+\frac{\alpha+1}{2}} = \emptyset \). By Lemma 3.4,
\[
|X_{i-1} \cup X_i \cup X_{i+1}| \geq 2\delta + 2 \quad \text{for all } i \quad \left( 3 \leq i \leq 3k + \frac{\alpha-3}{2} \right).
\]

Claim 4.1  
(i) We have \( |\bigcup_{5 \leq i \leq 13} X_i| < 7(\delta + 1) \).

(ii) For \( i_0 \in \{6, 9\} \), \( |\bigcup_{i_0-1 \leq i \leq i_0+1} X_i| < 3(\delta + 1) \) and \( |\bigcup_{i_0-1 \leq i \leq i_0+4} X_i| < 5(\delta + 1) \).

Proof. We first show that (i) holds. Suppose that \( |\bigcup_{5 \leq i \leq 13} X_i| \geq 7(\delta + 1) \). It follows from (4.1) that if \( \alpha = 3 \), then
\[
|V(G)| \geq |N_G[a_0]| + \sum_{\substack{1 \leq j \leq k \\{2,3,4\}}} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}| + \left| \bigcup_{5 \leq i \leq 13} X_i \right| \\
\geq (\delta + 1) + 2(k-3)(\delta + 1) + 7(\delta + 1) \\
> \left( 2k + \frac{5}{3} \right)(\delta + 1);
\]
if $\alpha = 5$, then for $z \in X_{3k+3}$,

$$|V(G)| \geq |N_G[a_0]| + \sum_{1 \leq j \leq k} |X_{3j-1} \cup X_{3j} \cup X_{3j+1}| + \bigcup_{5 \leq i \leq 13} X_i | + |N_G[z]|$$

$$\geq (\delta + 1) + 2(k - 3)(\delta + 1) + 7(\delta + 1) + (\delta + 1)$$

$$> \left(2k + \frac{7}{3}\right)(\delta + 1).$$

In either case, we obtain a contradiction. Thus (i) holds.

For $i_0 \in \{6, 9\}$, if $|\bigcup_{i_0-1 \leq i \leq i_0+1} X_i| \geq 3(\delta + 1)$ or $|\bigcup_{i_0-1 \leq i \leq i_0+4} X_i| \geq 5(\delta + 1)$, then by (4.1), we have $|\bigcup_{5 \leq i \leq 13} X_i| \geq 7(\delta + 1)$, which contradicts (i). Thus (ii) holds. ■

For each $i \in \{6, 9\}$, let $H_i$ be the graph on $X_i$ defined by

$$E(H_i) = \{xy : x, y \in X_i, 1 \leq d_G(x, y) \leq 2\},$$

and let $\alpha(H_i)$ denote the independence number of $H_i$, i.e., the maximum cardinality of a subset of $X_i$ which is independent in $H_i$.

**Claim 4.2** For each $i_0 \in \{6, 9\}$, $\alpha(H_{i_0}) \leq 2$.

**Proof.** Suppose that $\alpha(H_{i_0}) \geq 3$. Then for an independent set $\{x_1, x_2, x_3\}$ of $H_{i_0}$, $N_G[x_1]$, $N_G[x_2]$ and $N_G[x_3]$ are pairwise disjoint subsets of $X_{i_0-1} \cup X_{i_0} \cup X_{i_0+1}$. Hence

$$|X_{i_0-1} \cup X_{i_0} \cup X_{i_0+1}| \geq |N_G[x_1]| + |N_G[x_2]| + |N_G[x_3]| \geq 3(\delta + 1),$$

which contradicts Claim 4.1(ii). ■

**Claim 4.3** For each $i_0 \in \{6, 9\}$, $H_{i_0}$ consists of exactly two complete components.

**Proof.** Suppose that $H_{i_0}$ is connected. It is known that if a connected graph $H$ satisfies $\alpha(H) \leq 2$, then rad$(H) \leq 2$. Thus by Claim 4.2, rad$(H_{i_0}) \leq 2$. Let $c$ be a central vertex of $H_{i_0}$. Then for every vertex $x \in X_{i_0}$, we have $d_G(c, x) \leq 2d_{H_{i_0}}(c, x) \leq 4$. Note that $4 + 1 \leq i_0 \leq 3k + \frac{\alpha + 1}{2} - 4$ because $k \geq 6$ and $\alpha \in \{3, 5\}$. Hence, applying Lemma 3.3 with $h = 4$, we see that $G$ has a $d$-good set, which is a contradiction. Thus $H_{i_0}$ is disconnected. This together with Claim 4.2 leads to the desired conclusion. ■

For each $i_0 \in \{6, 9\}$, let $A_{i_0}$ and $B_{i_0}$ be disjoint subsets of $X_{i_0}$ such that $H_{i_0}$ consists of two complete components $H_{i_0}[A_{i_0}]$ and $H_{i_0}[B_{i_0}]$. By the definition of $H_{i_0}$,
\(X_{i_0} = A_{i_0} \cup B_{i_0}\) and

- for \(x, x' \in A_{i_0}\) and \(y, y' \in B_{i_0}\), \(d_G(x, x') \leq 2\), \(d_G(y, y') \leq 2\) and \(d_G(x, y) \geq 3\).

For \(i_0 \in \{6, 9\}\) and \(s \in \{0, 1, 2, 3\}\), set \(A^s_{i_0} = N^s_G(A_{i_0}) \cap X_{i_0+s}\) and \(B^s_{i_0} = N^s_G(B_{i_0}) \cap X_{i_0+s}\). Note that \(A^0_{i_0} = A_{i_0}, B^0_{i_0} = B_{i_0}\) and \(X_{i_0+s} = A^s_{i_0} \cup B^s_{i_0}\).

**Claim 4.4** Let \(i_0 \in \{6, 9\}\), and let \(s, t \in \{0, 1, 2\}\) be integers with \((s, t) \neq (2, 2)\). Then \(d_G(A^s_{i_0}, B^t_{i_0}) \geq 3\).

**Proof.** Suppose that there exist \(a \in A^s_{i_0}\) and \(b \in B^t_{i_0}\) such that \(d_G(a, b) \leq 2\). Let \(a_{i_0} \in A_{i_0}\) and \(b_{i_0} \in B_{i_0}\) be vertices such that \(d_G(a, a_{i_0}) = s\) and \(d_G(b, b_{i_0}) = t\). Without loss of generality, we may assume that \(s \leq t\). Then \(s \leq 1\).

Let \(x \in X_{i_0+t}\). If \(x \in A^t_{i_0}\), then for a vertex \(a'_{i_0} \in A_{i_0}\) with \(d_G(a'_{i_0}, x) = t\),

\[d_G(b, x) = d_G(b, a) + d_G(a, a_{i_0}) + d_G(a_{i_0}, a'_{i_0}) + d_G(a'_{i_0}, x) \leq 2 + s + 2 + t \leq t + 5;\]

if \(x \in B^t_{i_0}\), then for a vertex \(b'_{i_0} \in B_{i_0}\) with \(d_G(b'_{i_0}, x) = t\),

\[d_G(b, x) = d_G(b, b_{i_0}) + d_G(b_{i_0}, b'_{i_0}) + d_G(b'_{i_0}, x) \leq t + 2 + t \leq t + 4.\]

In either case, we have \(d_G(b, x) \leq t + 5\). Since \(t + 6 \leq i_0 + t \leq 3k + \frac{n+1}{2} - (t + 5)\), applying Lemma 3.3 with \(h = t + 5\), we see that \(G\) has a \(d\)-good set, which is a contradiction. \(\blacksquare\)

**Claim 4.5** For \(i_0 \in \{6, 9\}\), \(d_G(A^2_{i_0}, B^2_{i_0}) \geq 3\).

**Proof.** Suppose that there exist \(a \in A^2_{i_0}\) and \(b \in B^2_{i_0}\) such that \(d_G(a, b) \leq 2\). Let \(aa_{i_0+1}a_{i_0}\) be a shortest path joining \(a\) and \(A_{i_0}\), and let \(bb_{i_0+1}b_{i_0}\) be a shortest path joining \(b\) and \(B_{i_0}\). Note that \(a_{i_0+1} \in A^1_{i_0}\) and \(b_{i_0+1} \in B^1_{i_0}\).

If \(d_G(a, b) \leq 1\), then \(d_G(a_{i_0+1}, b) \leq d_G(a_{i_0+1}, a) + d_G(a, b) \leq 1 + 1\), which contradicts Claim 4.4. Thus \(d_G(a, b) = 2\) (i.e., \(a \neq b\), \(ab \notin E(G)\) and \(N_G(a) \cap N_G(b) \neq \emptyset\)). Let \(c \in N_G(a) \cap N_G(b)\). Note that \(c \in X_{i_0+1} \cup X_{i_0+2} \cup X_{i_0+3}\). Suppose that \(c \in X_{i_0+1} \cup X_{i_0+2}\). Without loss of generality, we may assume that \(c \in A^1_{i_0} \cup A^2_{i_0}\). Then for \(c^* \in N_G[c] \cap A^1_{i_0}\), \(d_G(c^*, b) \leq d_G(c^*, c) + d_G(c, b) \leq 1 + 1\), which contradicts Claim 4.4. Thus \(c \in X_{i_0+3}\).

**Subclaim 4.5.1** One of the following holds:

(i) for every \(x \in A^2_{i_0}\), \(d_G(c, x) \leq 6\); or

(ii) for every \(y \in B^2_{i_0}\), \(d_G(c, y) \leq 6\).
Claim 4.4, (4.2) and (4.3), these sets are pairwise disjoint. Since they are subsets of Claim 4.1(ii).

Subclaim 4.6.1 Suppose that there exist $x_{i_0+2} \in A^2_{i_0}$ and $y_{i_0+2} \in B^2_{i_0}$ such that $d_G(c, x_{i_0+2}) \geq 7$ and $d_G(c, y_{i_0+2}) \geq 7$. Let $x_{i_0+2}a'_{i_0+1}a_0$ be a shortest path joining $x_{i_0+2}$ and $A_{i_0}$, and let $y_{i_0+2}b'_{i_0+1}b_0$ be a shortest path joining $y_{i_0+2}$ and $B_{i_0}$. Note that $a'_{i_0+1} \in A^1_{i_0}$ and $b'_{i_0+1} \in B^1_{i_0}$. If $d_G(c, a'_{i_0+1}) \leq 2$, then

$$d_G(c, x_{i_0+2}) = d_G(c, a'_i) + d_G(a'_{i_0+1}, x_{i_0+2}) \leq 2 + 1;$$

if $d_G(a_{i_0}, a'_{i_0+1}) \leq 2$, then

$$d_G(c, x_{i_0+2}) = d_G(c, a) + d_G(a, a_{i_0}) + d_G(a_{i_0}, a'_{i_0+1}) + d_G(a'_{i_0+1}, x_{i_0+2}) \leq 1 + 2 + 2 + 1.$$

In either case, we obtain a contradiction. Thus

$$d_G(c, a'_{i_0+1}) \geq 3 \quad \text{and} \quad d_G(a_{i_0}, a'_{i_0+1}) \geq 3. \quad (4.2)$$

By a similar argument, we also get

$$d_G(c, b'_{i_0+1}) \geq 3 \quad \text{and} \quad d_G(b_{i_0}, b'_{i_0+1}) \geq 3. \quad (4.3)$$

Now we consider five sets $N_G[c], N_G[a_{i_0}], N_G[a'_{i_0+1}], N_G[b_{i_0}], N_G[b'_{i_0+1}]$. By Claim 4.4, (4.2) and (4.3), these sets are pairwise disjoint. Since they are subsets of $\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}$, it follows that $|\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}| \geq 5(\delta + 1)$, which contradicts Claim 4.1(ii).

Considering Subclaim 4.5.1, without loss of generality, we may assume that $d_G(c, x) \leq 6$ for all $x \in A^2_{i_0}$. This implies that $d_G(b, x) \leq d_G(b, c) + d_G(c, x) \leq 7$ for all $x \in A^2_{i_0}$. Let $y \in B^2_{i_0}$. Then for a vertex $b' \in B_{i_0}$ with $d_G(b', y) = 2$,

$$d_G(b, y) \leq d_G(b, b_{i_0}) + d_G(b_{i_0}, b') + d_G(b', y) \leq 2 + 2 + 2.$$

Consequently, $d_G(b, x) \leq 7$ for all $x \in X_{i_0+2}$. Since $7 + 1 \leq i_0 + 2 \leq 3k + \frac{a+1}{2} - 7$, applying Lemma 3.3 with $h = 7$, we see that $G$ has a $d$-good set, which is a contradiction.

Claim 4.6 For $i_0 \in \{6, 9\}$, $d_G(A^3_{i_0}, B^3_{i_0}) \geq 3$.

Proof. Suppose that there exist $a \in A^3_{i_0}$ and $b \in B^3_{i_0}$ such that $d_G(a, b) \leq 2$. Let $aa_{i_0}a_{i_0+1}a_{i_0}$ be a shortest path joining $a$ and $A_{i_0}$, and let $bb_{i_0}b_{i_0+1}b_{i_0}$ be a shortest path joining $b$ and $B_{i_0}$. We first prove the following subclaim.

Subclaim 4.6.1 For $x \in A^3_{i_0}$, we have $d_G(b, x) \leq 8$. 
Proof. By way of contradiction, suppose that there exists a vertex $x_{i_0+3} \in A_{i_0}^3$ such that $d_G(b, x_{i_0+3}) \geq 9$. Then $d_G(a, x_{i_0+3}) \geq 7$. Let $x_{i_0+3}a_{i_0+2}a_{i_0+1}a_{i_0}$ be a shortest path joining $x_{i_0+3}$ and $A_{i_0}$. If $d_G(a_{i_0+2}, x_{i_0+3}) \leq 2$, then
\[ d_G(a, x_{i_0+3}) = d_G(a, a_{i_0+2}) + d_G(a_{i_0+2}, x_{i_0+3}) \leq 1 + 2; \]
if $d_G(a_{i_0+2}, a_{i_0}') \leq 2$, then
\[ d_G(a, x_{i_0+3}) = d_G(a, a_{i_0+2}) + d_G(a_{i_0+2}, a_{i_0}') + d_G(a_{i_0}', x_{i_0+3}) \leq 1 + 2 + 3. \]
In either case, we obtain a contradiction. Thus
\[ d_G(a_{i_0+2}, x_{i_0+3}) \geq 3 \] and $d_G(a_{i_0+2}, a_{i_0}') \geq 3$. (4.4)

Suppose that $d_G(b, a_{i_0+2}) \geq 3$. Now we consider five sets $N_G[x_{i_0+3}], N_G[a_{i_0+2}], N_G[a_{i_0}'], N_G[b], N_G[b_{i_0}]$. Recall that $d_G(b, x_{i_0+3}) \geq 9$. Hence by Claim 4.4 and (4.4), these sets are pairwise disjoint. Since they are subsets of $\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}$, it follows that $|\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}| \geq 5(\delta + 1)$, which contradicts Claim 4.1(ii). Thus $d_G(b, a_{i_0+2}) \leq 2$.

If $d_G(a_{i_0+1}, x_{i_0+3}) \leq 2$, then
\[ d_G(a, x_{i_0+3}) = d_G(a, a_{i_0+1}) + d_G(a_{i_0+1}, x_{i_0+3}) \leq 2 + 2; \]
if $d_G(a_{i_0+1}, a_{i_0}') \leq 2$, then
\[ d_G(b, x_{i_0+3}) = d_G(b, a_{i_0+2}) + d_G(a_{i_0+2}, a_{i_0+1}) + d_G(a_{i_0+1}, a_{i_0}') + d_G(a_{i_0}', x_{i_0+3}) \]
\[ \leq 2 + 1 + 2 + 3; \]
if $d_G(b, a_{i_0+1}) \leq 2$, then
\[ d_G(b, x_{i_0+3}) = d_G(b, a_{i_0+1}) + d_G(a_{i_0+1}, a_{i_0}) + d_G(a_{i_0}, a_{i_0}') + d_G(a_{i_0}', x_{i_0+3}) \]
\[ \leq 2 + 1 + 2 + 3. \]
In any case, we obtain a contradiction. Thus
\[ d_G(a_{i_0+1}, x_{i_0+3}) \geq 3, \] $d_G(a_{i_0+1}, a_{i_0}') \geq 3$ and $d_G(b, a_{i_0+1}) \geq 3$. (4.5)

Now we consider five sets $N_G[x_{i_0+3}], N_G[a_{i_0+1}], N_G[a_{i_0}'], N_G[b], N_G[b_{i_0}]$. By Claim 4.4 and (4.5), these sets are pairwise disjoint. Since they are subsets of $\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}$, it follows that $|\bigcup_{i_0-1 \leq i' \leq i_0+4} X_{i'}| \geq 5(\delta + 1)$, which contradicts Claim 4.1(ii). \[ \blacksquare \]

We show that $d_G(b, x) \leq 8$ for all $x \in X_{i_0+3}$. If $x \in A_{i_0}^3$, then by Subclaim 4.6.1, we have $d_G(b, x) \leq 8$. Thus we may assume that $x \in B_{i_0}^3$. Let $b_{i_0}' \in B_{i_0}$ be a vertex with $d_G(b_{i_0}', x) = 3$. Then
\[ d_G(b, x) \leq d_G(b, b_{i_0}) + d_G(b_{i_0}, b_{i_0}') + d_G(b_{i_0}', x) \leq 3 + 2 + 3. \]
Consequently, \( d_G(b, x) \leq 8 \) for all \( x \in X_{i_0+3} \). Since \( 8 + 1 \leq i_0 + 3 \leq 3k + \frac{2k+1}{2} - 8 \), applying Lemma 3.3 with \( h = 8 \), we see that \( G \) has a \( d \)-good set, which is a contradiction. ■

**Claim 4.7** For \( i_0 \in \{6, 9\} \), \( d_G(\bigcup_{0 \leq s \leq 3} A_{i_0}^s, \bigcup_{0 \leq t \leq 3} B_{i_0}^t) \geq 3 \).

*Proof.* Assume \((s, t) \notin \{(1, 3), (3, 1), (2, 3), (3, 2)\} \) then by Claims 4.4–4.6, \( d_G(A_{i_0}^s, B_{i_0}^t) \geq 3 \). Thus, by symmetry, it suffices to show that \( d_G(A_{i_0}^1, A_{i_0}^2, B_{i_0}^3) \geq 3 \). Suppose that there exist \( a \in A_{i_0}^1 \cup A_{i_0}^2 \) and \( b \in B_{i_0}^3 \) such that \( d_G(a, b) \leq 2 \) (i.e., \( N_G(a) \cap N_G(b) \neq \emptyset \)). Let \( c \in N_G(a) \cap N_G(b) \). Note that \( c \in A_{i_0}^2 \cup B_{i_0}^3 \cup A_{i_0}^3 \cup B_{i_0}^3 \). By Claim 4.6, \( c \notin A_{i_0}^4 \).

If \( c \in A_{i_0}^2 \), then for a vertex \( b' \in N_G(c) \cap B_{i_0}^2 \), \( d_G(c, b') \leq d_G(c, b) + d_G(b, b') = 2 \), which contradicts Claim 4.5. Thus \( c \in B_{i_0}^2 \cup B_{i_0}^3 \). Then for a vertex \( c' \in N_G(c) \cap B_{i_0}^2 \), \( d_G(a, c') = d_G(a, c) + d_G(c, c') \leq 2 \), which contradicts Claim 4.4 or 4.5. ■

Considering Claim 4.6, without loss of generality, we may assume that \( A_{i_0}^3 = A_9 \) and \( B_{i_0}^3 = B_9 \). Let \( A'_i := A_{i_0}^{i-6} \) and \( B'_i := B_{i_0}^{i-6} \) for each \( i \) \((6 \leq i \leq 9)\) and let \( A'_i := A_{i_0}^{i-9} \) and \( B'_i := B_{i_0}^{i-9} \) for each \( i \) \((10 \leq i \leq 12)\).

**Claim 4.8** We have \( d_G(\bigcup_{6 \leq i \leq 12} A'_i, \bigcup_{6 \leq i \leq 12} B'_i) \geq 3 \).

*Proof.* Suppose that \( d_G(\bigcup_{6 \leq i \leq 12} A'_i, \bigcup_{6 \leq i \leq 12} B'_i) \leq 2 \). Then by Claim 4.7, \( d_G(A_{6'}^{10}, B_{6'}^{10}) = 2 \) or \( d_G(A_{10}^{10}, B_{6'}^{10}) = 2 \). Without loss of generality, we may assume that there exist \( a \in A_{6'} \) \((= A_{9}^{0})\) and \( b \in B_{10}^{10} \) \((= B_{9}^{0})\) such that \( d_G(a, b) = 2 \). Let \( c \in N_G(a) \cap N_G(b) \). Note that \( c \in X_9 \). If \( c \in A_{6'} \) \((= A_{9}^{0})\), then \( d_G(A_{6'}^{10}, B_{9}^{10}) = d_G(c, b) = 1 \); if \( c \in B_{10}^{10} \) \((= B_{9}^{0})\), then \( d_G(A_{6'}^{10}, B_{9}^{10}) = d_G(a, c) = 1 \). In either case, we obtain a contradiction. ■

**Claim 4.9** There exists an index \( i_0 \in \{6, 8, 10\} \) and there exists a vertex \( a_{i_0} \in A_{i_0}^\prime \) such that \( d_G(a_{i_0}, x) = 2 \) for all \( x \in A_{i_0+2}^\prime \).

*Proof.* By way of contradiction, suppose that the claim does not hold. We recursively define vertices \( a_i \in A_i^\prime \) \((i \in \{6, 8, 10, 12\})\) as follows: Let \( a_6 \in A_6^\prime \). For each \( i \in \{6, 8, 10\} \), by the assumption that the claim does not hold, there exists a vertex \( a_{i+2} \in A_{i+2}^\prime \) such that \( d_G(a_i, a_{i+2}) \geq 3 \). Then \( N_G(a_i) \) \((i \in \{6, 8, 10, 12\})\) are pairwise disjoint. For each \( j \in \{6, 9, 12\} \), let \( b_j \in B_j^\prime \). Then by Claim 4.8, \( N_G(a_i) \) \((i \in \{6, 8, 10, 12\})\) and \( N_G(b_j) \) \((j \in \{6, 9, 12\})\) are pairwise disjoint. Since these sets are subsets of \( \bigcup_{5 \leq i \leq 13} X_i \), it follows that \(|\bigcup_{5 \leq i \leq 13} X_i| \geq 7(\delta + 1)\), which contradicts Claim 4.1(i). ■

By symmetry, we obtain the following claim.
Claim 4.10 For some index \( j_0 \in \{6, 8, 10\} \), there exists a vertex \( b_{j_0} \in B'_{j_0} \) such that \( d_G(b_{j_0}, y) = 2 \) for all \( y \in B'_{j_0+2} \).

Let \( a_{i_0} \) and \( b_{j_0} \) as in Claims 4.9 and 4.10, respectively. Let \( a_0a_1^*a_2^* \cdots a_{i_0}^* \) be a shortest path joining \( a_0 \) and \( a_{i_0} \), where \( a_{i_0}^* = a_{i_0} \), and let \( a_0b_1^*b_2^* \cdots b_{j_0}^* \) be a shortest path joining \( a_0 \) and \( b_{j_0} \), where \( b_{j_0}^* = b_{j_0} \).

Fix an index \( l \left( 3k + \frac{a-3}{2} \leq l \leq 3k + \frac{a+1}{2} \right) \) and a vertex \( x \in X_l \). Then there exists a vertex \( y \in X_{12} \) such that \( d_G(y, x) = l - 12 \). Note that either \( y \in A'_{12} \) or \( y \in B'_{12} \).

If \( y \in A'_{12} \), then there exists a vertex \( y' \in A'_{i_0+2} \) such that \( d_G(y', y) = 12 - (i_0 + 2) = 10 - i_0 \), and hence
\[
d_G(a_0^*, x) \leq d_G(a_0^*, a_{i_0}^*) + d_G(a_{i_0}^*, y') + d_G(y', y) + d_G(y, x) \\
= (i_0 - 2) + 2 + (10 - i_0) + (l - 12) \\
= l - 2;
\]

If \( y \in B'_{12} \), then there exists a vertex \( y' \in B'_{j_0+2} \) such that \( d_G(y', y) = 12 - (j_0 + 2) = 10 - j_0 \), and hence
\[
d_G(b_{j_0}^*, x) \leq d_G(b_{j_0}^*, b_{j_0}) + d_G(b_{j_0}, y') + d_G(y', y) + d_G(y, x) \\
= (j_0 - 2) + 2 + (10 - j_0) + (l - 12) \\
= l - 2.
\]

In either case, we have \( d_G(\{a_0^*, b_{j_0}^*\}, x) \leq l - 2 \). Therefore by Lemma 3.6, \( G \) has a \( d \)-good set, which is a contradiction.

This completes the proof of Theorem 1.1.

5 The case \( d \in \{5, 9\} \)

In this section, we prove Theorem 1.2, dividing the proof into two cases. We first consider the case where \( d = 5 \).

Theorem 5.1 Let \( G \) be a connected graph. If \( |V(G)| < \frac{7(\delta(G)+1)}{3} \), then there exists a spanning tree \( T \) of \( G \) such that \( \text{diam}(T) \leq 5 \).

Proof. Suppose that \( G \) has no spanning tree \( T \) such that \( \text{diam}(T) \leq 5 \). Then by Lemma 3.1, \( G \) has no 5-good set. In particular, \( \text{ecc}_G(x) \geq 3 \) for every \( x \in V(G) \), and hence there exists a path \( a_0a_1a_2a_3 \) such that \( d_G(a_0, a_3) = 3 \).

Claim 5.1 For each \( i \in \{0, 2\} \), \( |N_G[a_i] \cap N_G[a_{i+1}]| > \frac{2(\delta(G)+1)}{3} \).
Proof. Since \( \{a_i, a_{i+1}\} \) is not 5-good, there exists a vertex \( x \in V(G) \) such that 
\[
d_G(\{a_i, a_{i+1}\}, x) \geq 3 \quad \text{(i.e., } (N_G[a_i] \cup N_G[a_{i+1}]) \cap N_G[x] = \emptyset).\]
Then
\[
|V(G)| \geq |N_G[a_i] \cup N_G[a_{i+1}] \cup N_G[x]|
= |N_G[a_i]| + |N_G[a_{i+1}]| - |N_G[a_i] \cap N_G[a_{i+1}]| + |N_G[x]|
\geq 3(\delta(G) + 1) - |N_G[a_i] \cap N_G[a_{i+1}]|.
\]
Since \( |V(G)| < \frac{7(\delta(G)+1)}{3} \), this leads to the desired inequality. ■

Since \( N_G[a_0] \cap N_G[a_3] = \emptyset \), it follows from Claim 5.1 that
\[
|(N_G[a_0] \cap N_G[a_1]) \cup (N_G[a_2] \cap N_G[a_3])| = |N_G[a_0] \cap N_G[a_1]| + |N_G[a_2] \cap N_G[a_3]|
\geq \frac{4(\delta(G) + 1)}{3}.
\]
(5.1)
Since \( \{a_1, a_2\} \) is not 5-good, there exists a vertex \( x \in V(G) \) such that 
\[
d_G(\{a_1, a_2\}, x) \geq 3 \quad \text{(i.e., } (N_G[a_1] \cup N_G[a_2]) \cap N_G[x] = \emptyset).\]
It follows from (5.1) that
\[
|V(G)| \geq |(N_G[a_0] \cap N_G[a_1]) \cup (N_G[a_2] \cap N_G[a_3]) \cup N_G[x]|
= |(N_G[a_0] \cap N_G[a_1]) \cup (N_G[a_2] \cap N_G[a_3])| + |N_G[x]|
\geq \frac{4(\delta(G) + 1)}{3} + (\delta(G) + 1)
= \frac{7(\delta(G) + 1)}{3},
\]
which contradicts the assumption of the theorem. ■

Next we prove Theorem 1.2 for the case where \( d = 9 \).

Theorem 5.2 Let \( G \) be a connected graph. If \(|V(G)| < \frac{11(\delta(G)+1)}{4}\), then there exists a spanning tree \( T \) of \( G \) such that \( \text{diam}(T) \leq 9 \).

Proof. Suppose that \( G \) has no spanning tree \( T \) such that \( \text{diam}(T) \leq 9 \). Then by Lemma 3.1, \( G \) has no 9-good set.

Claim 5.2 There exist no four vertices \( u_1, \ldots, u_4 \in V(G) \) such that \( N_G[u_i] \) (1 \( \leq \) \( i \leq 4 \)) are pairwise disjoint.

Proof. If \( N_G[u_i] \) (1 \( \leq \) \( i \leq 4 \)) are pairwise disjoint for four vertices \( u_1, \ldots, u_4 \in V(G) \), then
\[
\frac{11(\delta(G) + 1)}{3} \geq |V(G)|
\geq 4(\delta(G) + 1),
\]
which is a contradiction. ■
Claim 5.3 For \( a \in V(G) \), \( N_G^6(a) = \emptyset \).

Proof. Suppose that \( N_G^6(a) \neq \emptyset \), and let \( x \in N_G^6(a) \). Let \( b_0b_1 \cdots b_6 \) be a path joining \( a \) and \( x \), where \( b_0 = a \) and \( b_6 = x \). For \( i \in \{2, 4\} \), since \( \{b_3, b_i\} \) is not 9-good, there exists a vertex \( y_i \in V(G) \) such that \( d_G(\{b_3, b_i\}, y_i) \geq 5 \). In particular, \((N_G[b_0] \cup N_G[b_3]) \cap N_G[y_2] = \emptyset \) and \((N_G[b_3] \cup N_G[b_6]) \cap N_G[y_4] = \emptyset \).

Since \( N_G[b_0], N_G[b_3] \) and \( N_G[b_6] \) are pairwise disjoint, it follows from Claim 5.2 that \( N_G[b_6] \cap N_G[y_2] \neq \emptyset \) and \( N_G[b_0] \cap N_G[y_4] \neq \emptyset \). Thus
\[
5 \leq d_G(b_3, y_2) \leq d_G(b_3, b_6) + d_G(b_6, y_2) \leq 3 + 2
\]
and
\[
5 \leq d_G(b_3, y_4) \leq d_G(b_3, b_0) + d_G(b_0, y_4) \leq 3 + 2.
\]
This forces \( d_G(b_6, y_2) = d_G(b_0, y_4) = 2 \). Hence there exist vertices \( y'_2 \in N_G(b_6) \cap N_G(y_2) \) and \( y'_4 \in N_G(b_0) \cap N_G(y_4) \).

If \((N_G[b_3] \cup N_G[b_4]) \cap N_G[y_4] \neq \emptyset \), then we easily verify that \( d_G(\{b_3, b_4\}, y_4) \leq 4 \), which is a contradiction. Thus \((N_G[b_1] \cup N_G[b_4]) \cap N_G[y_4] = \emptyset \). This together with the fact that \( d_G(b_1, b_4) = 3 \) implies that \( N_G[b_1], N_G[b_4] \) and \( N_G[y_4] \) are pairwise disjoint. It follows from Claim 5.2 that \((N_G[b_1] \cup N_G[b_4] \cup N_G[y_4]) \cap N_G[y'_2] \neq \emptyset \). If \( N_G[b_1] \cap N_G[y'_2] \neq \emptyset \), then
\[
d_G(b_2, y_2) \leq d_G(b_2, b_1) + d_G(b_1, y'_2) + d_G(y'_2, y_2) \leq 1 + 2 + 1;
\]
if \( N_G[b_4] \cap N_G[y'_2] \neq \emptyset \), then
\[
d_G(b_3, y_2) \leq d_G(b_3, b_4) + d_G(b_4, y'_2) + d_G(y'_2, y_2) \leq 1 + 2 + 1.
\]
In either case, we obtain \( d_G(\{b_2, b_3\}, y_2) \leq 4 \), which contradicts the choice of \( y_2 \).
Thus \( N_G[y_4] \cap N_G[y'_2] \neq \emptyset \). Then
\[
d_G(b_0, b_6) \leq d_G(b_0, y_4) + d_G(y_4, y'_2) + d_G(y'_2, b_6) \leq 2 + 2 + 1,
\]
which is a contradiction. \( \blacksquare \)

Claim 5.4 Let \( u_0u_1 \cdots u_5 \) be a path such that \( d_G(u_0, u_5) = 5 \). Then \( |N_G[u_2] \cap N_G[u_3]| < \frac{2(\delta(G)+1)}{3} \).

Proof. Since \( \{u_2, u_3\} \) is not 9-good, there exists a vertex \( x \in V(G) \) such that \( d_G(\{u_2, u_3\}, x) \geq 5 \). For every \( i \ (0 \leq i \leq 5) \), since \( d_G(\{u_2, u_3\}, u_i) \leq 2 \), we have \( d_G(u_i, x) \geq 3 \). In particular,
\[
((N_G[u_2] \cap N_G[u_3]) \cup N_G[u_0] \cup N_G[u_5]) \cap N_G[x] = \emptyset.
\] (5.2)
If \(N_G[u_2] \cap N_G[u_3], N_G[u_0] \text{ and } N_G[u_5]\) are not pairwise disjoint, then we can verify that there exists a path of length at most 4 joining \(u_0\) and \(u_5\), which contradicts the fact that \(d_G(u_0, u_5) = 5\). Thus \(N_G[u_2] \cap N_G[u_3], N_G[u_0] \text{ and } N_G[u_5]\) are pairwise disjoint. This together with (5.2) implies that

\[
|V(G)| \geq |(N_G[u_2] \cap N_G[u_3]) \cup N_G[u_0] \cup N_G[u_5] \cup N_G[x]|
\geq |N_G[u_2] \cap N_G[u_3]| + 3(\delta(G) + 1).
\]

Since \(|V(G)| < \frac{11(\delta(G) + 1)}{3}\), it follows that the desired conclusion holds.  

Let \(a_0 \in V(G)\). Since \(\{a_0\}\) is not 9-good, \(ecc_G(a_0) \geq 5\). Hence there exists a path \(a_0a_1 \cdots a_5\) such that \(d_G(a_0, a_5) = 5\).

**Claim 5.5** For \(i \in \{0, 4\}, |N_G[a_i] \cap N_G[a_{i+1}]| < \frac{2(\delta(G) + 1)}{3}\).

**Proof.** By symmetry, it suffices to show that \(|N_G[a_0] \cap N_G[a_1]| < \frac{2(\delta(G) + 1)}{3}\). Since \(\{a_2, a_3\}\) is not 9-good, it follows from Claim 5.3 that there exists a vertex \(x \in V(G)\) such that \(d_G(a_2), x) = d_G(a_3, x) = 5\). Let \(a_3b_1 \cdots b_5\) be a shortest path joining \(a_3\) and \(x\), where \(b_5 = x\).

Since \(\{a_3, b_1\}\) is not 9-good, there exists a vertex \(z \in V(G)\) such that \(d_G(a_3, z) = d_G(b_1, z) = 5\). Then

\[
\left( \bigcup_{1 \leq i \leq 5} N_G[a_i] \right) \cup \left( \bigcup_{1 \leq i \leq 3} N_G[b_i] \right) \cap N_G[z] = \emptyset. \tag{5.3}
\]

If \(N_G[a_2] \cap N_G[b_3] \neq \emptyset\), then

\[
d_G(a_2, x) \leq d_G(a_2, b_3) + d_G(b_3, b_5) \leq 2 + 2,
\]

which is a contradiction. Thus \(N_G[a_2] \cap N_G[b_3] = \emptyset\).

If \(N_G[a_3] \cap N_G[b_3] = \emptyset\), then it follows from (5.3) that \(N_G[a_2], N_G[a_3], N_G[b_3]\) and \(N_G[z]\) are pairwise disjoint, which contradicts Claim 5.2. Thus \(N_G[a_5] \cap N_G[b_3] \neq \emptyset\).

If \(N_G[a_0] \cap N_G[b_3] \neq \emptyset\), then

\[
d_G(a_0, a_5) \leq d_G(a_0, b_3) + d_G(b_3, a_5) \leq 2 + 2,
\]

which contradicts the fact that \(d_G(a_0, a_5) = 5\). Thus \(N_G[a_0] \cap N_G[b_3] = \emptyset\). Consequently, \(N_G[a_0], N_G[a_3]\) and \(N_G[b_3]\) are pairwise disjoint. This together with Claim 5.2 and (5.3) implies that \(N_G[a_0] \cap N_G[z] \neq \emptyset\), and hence

\[
5 \leq d_G(a_3, z) \leq d_G(a_3, a_0) + d_G(a_0, z) \leq 3 + 2.
\]
This forces $d_G(a_0, z) = 2$. It follows that the path $a_3 a_2 a_1 a_0 z'$, where $z' \in N_G(a_0) \cap N_G(z)$, is a shortest path joining $a_3$ and $z$. Consequently it follows from Claim 5.4 that $|N_G[a_0] \cap N_G[a_1]| < \frac{2(\delta(G)+1)}{3}$.

By Claim 5.5, for $i \in \{0, 4\}$,

$$|N_G[a_i] \cup N_G[a_{i+1}]| = |N_G[a_i]| + |N_G[a_{i+1}]| - |N_G[a_i] \cap N_G[a_{i+1}]|
\geq 2(\delta(G)+1) - \frac{2(\delta(G)+1)}{3}
= \frac{4(\delta(G)+1)}{3}.$$ 

Since $d_G(a_0, a_5) = 5$, we have $(N_G[a_0] \cup N_G[a_1]) \cap (N_G[a_4] \cup N_G[a_5]) = \emptyset$.

Let $x \in V(G)$. If $(N_G[a_0] \cup N_G[a_1] \cup N_G[a_4] \cup N_G[a_5]) \cap N_G[x] = \emptyset$, then

$$|V(G)| \geq |N_G[a_0] \cup N_G[a_1] \cup N_G[a_4] \cup N_G[a_5] \cup N_G[x]|
= |N_G[a_0] \cup N_G[a_1]| + |N_G[a_4] \cup N_G[a_5]| + |N_G[x]|
\geq 2 \cdot \frac{4(\delta(G)+1)}{3} + (\delta(G)+1)
= \frac{11(\delta(G)+1)}{3},$$

which is a contradiction. Thus $(N_G[a_0] \cup N_G[a_1] \cup N_G[a_4] \cup N_G[a_5]) \cap N_G[x] \neq \emptyset$. Therefore we have $d_G(\{a_0, a_1\}, x) \leq 2$ or $d_G(\{a_4, a_5\}, x) \leq 2$, which leads to $d_G(\{a_2, a_3\}, x) \leq 4$. Since $x$ is arbitrary, $\{a_2, a_3\}$ is 9-good, which is a contradiction. This completes the proof of Theorem 5.2.

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**References**


